Plane and spherical harmonic representations of the geomagnetic field

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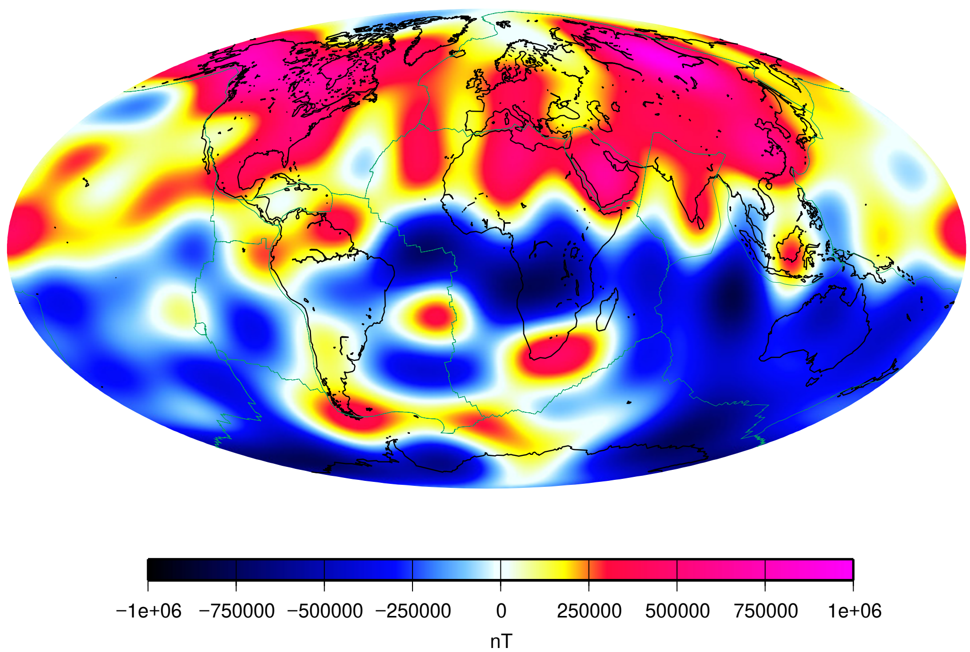

Figure 1: Vertical component of the main magnetic field at the core/mantle boundary up to spherical harmonic degree 12, model POMME-3.1

1 Harmonic representation of magnetic fields

Maxwell's equations for the magnetic field are

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J} + \partial_t \mathbf{D} && (1) \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (2) \end{aligned}$$

For a medium with isotropic magnetic permeability μ and electric permitivity ϵ, write $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ and the two equations are

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} &= \mu\mathbf{J} + \partial_t \mu\epsilon\mathbf{E} && (3) \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 && (4) \end{aligned}$$

Let us now find a convenient representation for the magnetic flux density **B**, which is simply called the magnetic field in geomagnetism. An arbitrary field $\mathbf{V}$ can always be written as the sum of the gradient of a scalar potential Ψ and the curl of a vector potential $\mathbf{A}$

$$\mathbf{V} = \boldsymbol{\nabla}\Psi + \boldsymbol{\nabla} \times \mathbf{A} \tag{5}$$

Note that this division into irrotational and non-divergent fields is generally not unique. There are 4 parameters on the right side (1 scalar and 3 vector components), while there are only 3 components of the vector field to be represented on the left side. Thus, we have some freedom in representing the field. The best choice for the magnetic field depends on the problem at hand: If the region of study contains sources of the magnetic field, a scalar potential will not help because the curl of $\boldsymbol{\nabla}\Psi$ is zero. On the other hand, we know that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Since the divergence of the curl is zero, a magnetic field represented by a vector potential $\mathbf{A}$ as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{6}$$

is a natural choice of representation in the presence of sources.

If there are no source currents in the region of study, such as when the internal magnetic field is modeled outside of the Earth, then the curl of **B** is zero. In this case, the magnetic field can be represented by the gradient of a scalar potential Ψ because the curl of the gradient of a scalar is zero.

Further considering that the divergence of B is zero, we find that this scalar potential Ψ has to fulfill the Laplace equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\Psi = 0 \tag{7}$$

which is also written as

$$\nabla^2\Psi = 0 \tag{8}$$

Functions Ψ which are solutions of the Laplace equation are called *harmonic*. In the following, we seek to represent the magnetic field by the gradient of harmonic functions. This will only work for that part of the magnetic field whose sources are outside of the study region. In case the sources are within the study region, one can represent the field by the vector potential **A**. However, there is a more convenient representation for this situation, by using poloidal and toroidal scalars, which will be discussed in a later lecture.

2 Solving the Laplace equation in Cartesian coordinates

Aproach (ansatz) by separation of variables:

$$T(x, y, z) = P(x)Q(y)R(z) \tag{9}$$

Filling into the Laplace equation gives

$$\begin{aligned} QR\partial_x^2 P + PR\partial_y^2 Q + PQ\partial_z^2 R &= 0 \quad (10) \\ \frac{\partial_x^2 P}{P} + \frac{\partial_y^2 Q}{Q} + \frac{\partial_z^2 R}{R} &= 0 \quad (11) \end{aligned}$$

Obviously, the best way to deal with this equation is to find those functions which will reproduce themselves under the operators ∂_x^2, ∂_y^2 and ∂_z^2. These Eigenfunctions are of the form

$$P(x) = P_0 e^{ik_x x},\ Q(y) = Q_0 e^{ik_y y},\ R(z) = R_0 e^{ik_z z} \tag{12}$$

with the *wavenumbers* k_x, k_y, and k_z. Filling the Eigenfunctions (12) into equation (11) gives

$$-k_x^2 - k_y^2 - k_z^2 = 0, \tag{13}$$

which means that the wavenumbers are not independent.
The usual situation in geomagnetism is that one has measurements in a plane at a given height. For example a marine or aeromagnetic survey. Then one would like to represent the field in the horizontal plane. The vertical continuation then follows from (13) as

$$k_z^2 = -k_x^2 - k_y^2, \rightarrow k_z = \pm i\sqrt{k_x^2 + k_y^2} \tag{14}$$

Thus, we can now define elementary solutions $U(k_x, k_y)$ for a given pair of horizontal wavenumbers k_x and k_y as

$$\begin{aligned} U(x, y, z) &= e^{ik_x x}\ e^{ik_y y}\ e^{ik_z z} \quad (15) \\ &= e^{ik_x x}\ e^{ik_y y}\ e^{\pm z\sqrt{k_x^2 + k_y^2}} \quad (16) \end{aligned}$$

Here, the term $\sqrt{k_x^2 + k_y^2}$ can be interpreted as the spatial frequency of a horizontal wave (Fig. 2). Its unit is *radians per distance*. There is no time dependence, so the wave does not propagate. However, we can still define the direction of the maximum variability, which is given by the vector (k_x, k_y). In the perpendicular direction $(-k_y, k_x)$ there is no variation at all.
What is the relevance of the $\pm$ in the exponent of the last term? Let us choose the z-axis pointing downward. Thus, z increases towards the Earth. For a positive exponent in the last term, $U(x, y, z)$ increases towards the Earth. Thus, this solution corresponds to source currents located below the plane. In contrast, for a negative sign in the exponent of the last term, $U(x, y, z)$ increases away from the Earth. This corresponds to source currents above the plane. These elementary functions can then be used to represent external sources, for example in the ionosphere.
When dealing with marine or aeromagnetic surveys, the external sources are usually not of interest. Instead, they are a source of noise which has to be removed. To a good approximation, the external sources can be considered as uniform throughout the survey area and only vary strongly in time. To correct for them in aeromagnetic surveys, one places one or more base stations in the survey area,

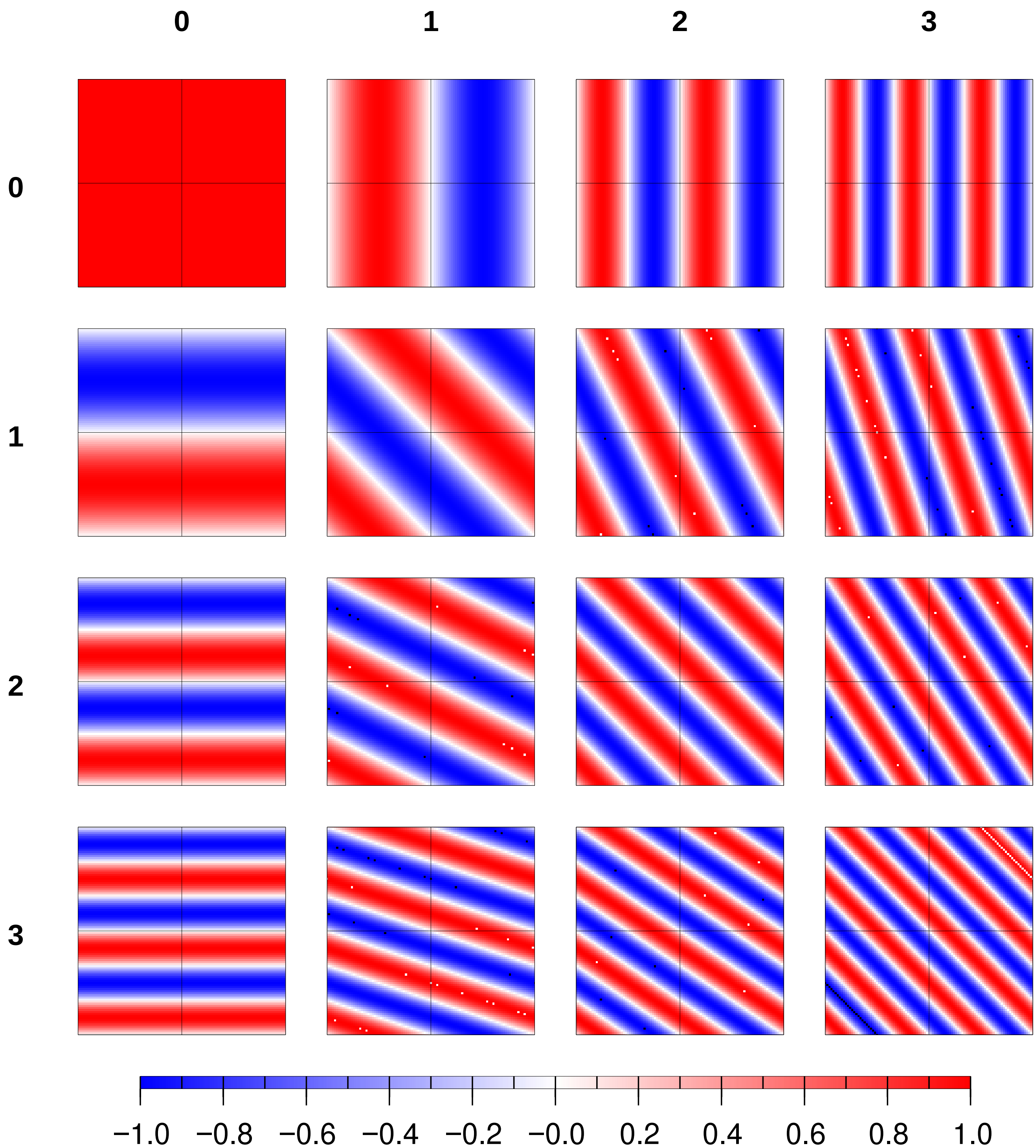

Figure 2: Harmonic basis functions in the plane for different pairs of the wavenumbers k_x and k_y

to record the time variations of the field. These variations are then subtracted from the aeromagnetic readings.
Afterward, one can assume that the sources of the field are internal to the Earth and represent the field

above the Earth by

$$T(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{T}(k_x,k_y) e^{ik_x x}\, e^{ik_y y}\, e^{z\sqrt{k_x^2+k_y^2}}\, dk_x\, dk_y \tag{17}$$

For the fixed altitude $z = 0$, $\tilde{T}(k_x, k_y)$ is the Fourier transform of $T(x, y)$. Thus, once the data has been gridded, it is easy represent the data by equation (17). In practice, x, y, k_x, and k_y take on discrete values defined by the grid cells. The corresponding discrete version of eq. (17) has a double sum instead of the double integral and looks rather messy.

While eq. (17) is for the magnetic potential, of course, magnetic field measurements do not directly yield the magnetic potential. It turns out, however, that the measured anomaly of the total intensity of the magnetic field also fulfills the Laplace equation. This holds true if the survey area is small enough to assume a constant direction of the main field (a few hundred km) and if the crustal field is small in comparison with the main field.

3 Solving the Laplace equation in 3D polar coordinates

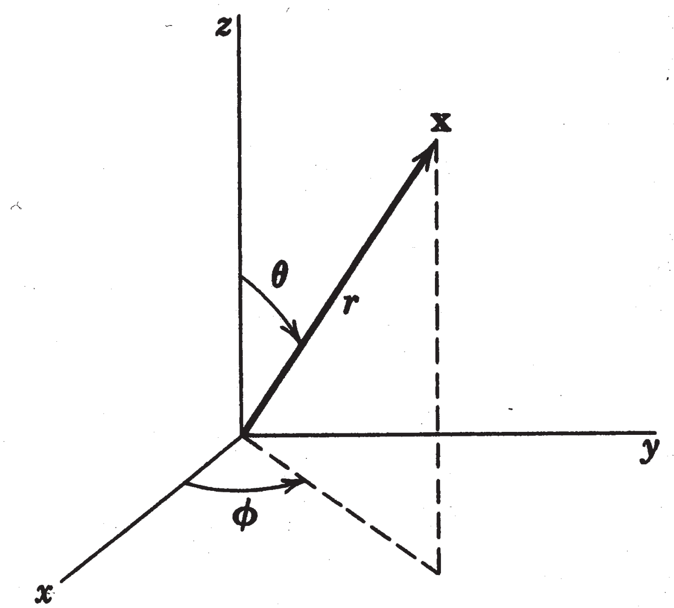

Figure 3: Spherical polar coordinates

For global field modeling, we use spherical coordinates as shown in Fig. 3. In polar coordinates, the Laplace operator can be written as

$$\nabla^2 = \frac{1}{r^2}(\partial_r r^2 \partial_r + \underbrace{\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta}_{\nabla_s^2}) \tag{18}$$

where ∇_s^2 is called the *surface* Laplace operator. After multiplying both sides with r^2, the Laplace equation can be written as

$$\partial_r r^2 \partial_r \Psi + \nabla_s^2 \Psi = 0 \tag{19}$$

We again solve the equation by separation of variables. In this case, we only separate into 2 functions, $Q(r)$ for the radial and $P(\theta, \phi)$ for the horizontal variation, respectively, as

$$\Psi(r, \theta, \phi) = Q(r)\ P(\theta, \phi) \tag{20}$$

Inserting ansatz (20) into equation (19) gives

$$P(\theta,\phi)\partial_r r^2 \partial_r Q(r) + Q(r)\nabla_s^2 P(\theta,\phi) \quad = \quad 0 \tag{21}$$

$$\frac{\partial_r r^2 \partial_r Q(r)}{Q(r)} + \frac{\nabla_s^2 P(\theta,\phi)}{P(\theta,\phi)} \quad = \quad 0 \tag{22}$$

This differential equation looks very similar to the plane case (11). Again, the trick is to find the Eigenfunctions of the differential operators. The Eigenfunctions of ∇_s^2 are the surface spherical harmonics $Y_\ell^m(\theta, \phi)$, which have the property

$$\nabla_s^2 Y_\ell^m = -\ell(\ell+1) Y_\ell^m \tag{23}$$

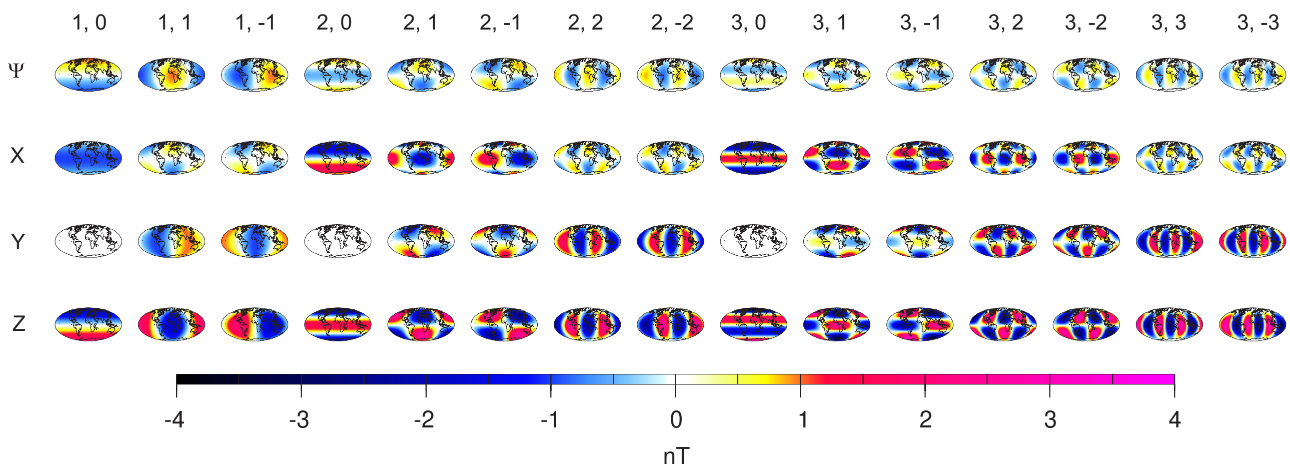

Table of Schmidt semi-normalized spherical harmonics up to degree 3. The potential, displayed in the top row, has been divided by the Earth radius in order to make its units compatible with the magnetic field element X, Y and Z.

Let us first take a look at these functions, which are displayed in Fig. 3. The index ℓ is called the degree. It specifies the total number of circles on which $Y_\ell^m(\theta, \phi)$ is zero. These zero-circles can either be in the meridional (North/South) direction, or they can be parallel to the equator.
The second index, m, is called the order. This is the number of circles in the meridional direction. They come in pairs, with a positive and a negative index, except when $m = 0$. The pair is identical, but one is rotated in longitude against the other. Obviously, the number of meridional zero-circles cannot exceed the total number of circles, so $|m| \leq \ell$.
The spherical harmonics with $m = 0$ have no variation in the longitudinal direction and are called *zonal* harmonics. The *sectorial* harmonics are those with $|m| = \ell$. They have no zero-lines parallel to the equator. Nevertheless, they still have a variation in the meridional direction. The mixed harmonics, which have both kinds of zero-circles, are called *tesseral*.
So, how are the indices ℓ and m related to the plane wavenumbers k_x and k_y? The answer is that ℓ can be interpreted as the spatial frequency. Precisely, the correspondence is

$$\frac{\sqrt{\ell(\ell+1)}}{r} = \sqrt{k_x^2 + k_y^2} \tag{24}$$

The index m, on the other hand, can be interpreted as a direction, or azimuth. At least in a qualitative sense, $\arcsin(m/\ell)$ corresponds to $\arctan(k_x/k_y)$.
Of course, these considerations about direction, and the division into zonal, tesseral and sectorial harmonics, depend on the definition of the coordinate system. For example, let us consider a zonal harmonic in one coordinate system. If the coordinate system is rotated, the same function will generally have to be represented using a sum of spherical harmonics of all orders. Even if the rotation was exactly 90 degrees. The only exceptions are for degree 0, which is just a constant, and for degree one. The three harmonics of degree 1 are equal and can be rotated into each other. For all higher harmonics this will not work. However, there is an interesting property regarding rotations: If a given function can be represented by spherical harmonics of a fixed degree ℓ in one coordinate system, then this is also possible in all other coordinate system orientations. Thus the spherical harmonic degree of a function is invariant under rotations, while its order depends on the orientation of the coordinate system.

Let us now use these spherical harmonics to solve the Laplace equation. Inserting the $Y_\ell^m(\theta,\phi)$ for $P(\theta,\phi)$ into equation (22) and using property (23) gives

$$\begin{aligned} \frac{\partial_r r^2 \partial_r Q(r)}{Q(r)} + \frac{\nabla_s^2 Y_\ell^m(\theta,\phi)}{Y_\ell^m(\theta,\phi)} &= 0 &(25)\\ \partial_r r^2 \partial_r Q(r) - \ell(\ell+1)Q(r) &= 0 &(26)\end{aligned}$$

The differential equation (26) has two kinds of solutions

$$\begin{aligned} Q_\ell(r) &= r^{-(\ell+1)} \quad \text{(internal sources)} &(27)\\ Q_\ell(r) &= r^\ell \quad \text{(external sources)} &(28)\end{aligned}$$

Here, the radial term $Q_\ell(r)$ depends only on the degree (spatial wavelength) and not on the order (direction) of the harmonic. In order to relate these functions to the radius of the Earth a and to arrive at convenient units, the internal solutions are multiplied with $a^{\ell+2}$ and the external ones with $a^{1-\ell}$. Combining with the horizontal dependence, the elementary solutions of the Laplace equation in spherical coordinates are given as

$$\begin{aligned} \Psi_\ell^m(r,\theta,\phi) &= a\left(\frac{a}{r}\right)^{\ell+1} Y_\ell^m(\theta,\phi) \quad \text{(internal sources)} &(29)\\ \Psi_\ell^m(r,\theta,\phi) &= a\left(\frac{r}{a}\right)^{\ell} Y_\ell^m(\theta,\phi) \quad \text{(external sources)} &(30)\end{aligned}$$

The potential of the magnetic field can then be represented as

$$\Psi(r,\theta,\phi) = a\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left[g_\ell^m\left(\frac{a}{r}\right)^{\ell+1} + k_\ell^m\left(\frac{r}{a}\right)^{\ell}\right] Y_\ell^m(\theta,\phi) \qquad (31)$$

and carries the unit $nT\ km$. Note that the $\ell = 0$ term is missing because there are no magnetic monopoles ($\mathbf{\nabla}\cdot\mathbf{B} = 0$). The coefficients g_ℓ^m and k_ℓ^m are called the Gauss coefficients of the field. These are the coefficients which are given in geomagnetic field models and which allow a user of the model to obtain the magnetic field vector at any desired location in the region between the external and internal sources of the field.

The vector of the magnetic field then follows from

$$\begin{aligned} \mathbf{B} &= -\mathbf{\nabla}\Psi &(32)\\ &= -(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta)\Psi &(33)\end{aligned}$$

For the vector components, as defined in Figure 4, we get

$$\begin{aligned} X(r,\theta,\phi) &= -B_\theta(r,\theta,\phi) = \frac{1}{r}\partial_\theta\Psi(r,\theta,\phi) &(34)\\ &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left[g_\ell^m\left(\frac{a}{r}\right)^{\ell+2} + k_\ell^m\left(\frac{r}{a}\right)^{\ell-1}\right]\partial_\theta Y_\ell^m(\theta,\phi) &(35)\\ Y(r,\theta,\phi) &= B_\phi(r,\theta,\phi) = -\frac{1}{r\sin\theta}\partial_\phi\Psi(r,\theta,\phi) &(36)\\ &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left[-g_\ell^m\left(\frac{a}{r}\right)^{\ell+2} - k_\ell^m\left(\frac{r}{a}\right)^{\ell-1}\right]\frac{1}{\sin\theta}\partial_\phi Y_\ell^m(\theta,\phi) &(37)\\ Z(r,\theta,\phi) &= -B_r(r,\theta,\phi) = \partial_r\Psi(r,\theta,\phi) &(38)\\ &= \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left[-(\ell+1)g_\ell^m\left(\frac{a}{r}\right)^{\ell+2} + \ell k_\ell^m\left(\frac{r}{a}\right)^{\ell-1}\right]Y_\ell^m(\theta,\phi) &(39)\end{aligned}$$

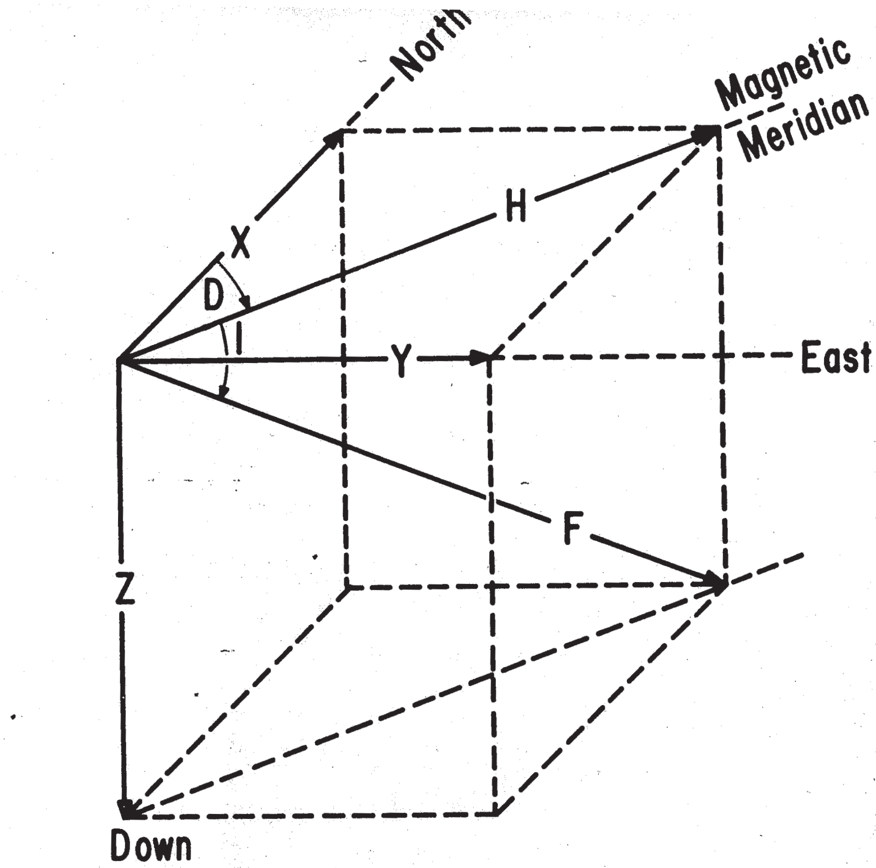

Figure 4: Definition of the geomagnetic field components used in geomagnetism: Inclination I, Declination D, Total intensity F, North component X, East component Y, and Doown component Z [*Langel*, 1987]

Thus, we can get the full vector of the magnetic field from a simple set of spherical harmonic coefficients of a scalar magnetic potential. As pointed out before, however, contributions to the magnetic field by electric currents passing through the region of interest have to be represented in a different way.

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